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A generalized positive energy theorem for spaces with asymptotic SUSY compactification

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Abstract

In this short note, we prove a generalized positive energy theorem for spaces with asymptotic SUSY compactification involving non-symmetric data. This work is motivated by the work of Dai [A positive mass theorem for spaces with asymptotic SUSY compactification, Comm. Math. Phys. 244 (2004) 335–345; A note on positive energy theorem for spaces with asymptotic SUSY compactification, 2004. arXiv:math-ph/0406006], Hertog–Horowitz–Maeda [Negative energy density in Calabi–Yau compactifications, JHEP 0305 (2003) 060], and Zhang [Angular momentum and positive mass theorem, Comm. Math. Phys. 206 (1999) 137–155].

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1. Introduction and statement of the result

In 1960, Arnowitt–Deser–Misner made a detailed study of isolated gravitational systems from the Hamiltonian point of view [1]. They discovered a conserved quantity given precisely by an integral and they concluded that this conserved quantity is the total energy

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of this isolated system. Mathematically rigorous proof of the conjecture that the total energy for asymptotically flat spaces is non-negative was firstly given by Schoen and Yau [10–12]. Shortly thereafter, Witten raised a simple proof using spinors from 'spacetime' view [14,9]. Later, various results have been established: Bartnik [2] defined the ADM mass for higher dimensional spin manifolds and generalized this theorem to that case; Zhang [16] globally defined the concept of angular momentum and proved a positive mass theorem involving this non-symmetric data which gave an answer to the 120th problem of Yau in his problem section [15].

In string theory [3], our universe is modelled by a 10-dimensional manifold which asymptotically approaches the product of a flat Minkowski space $M^{3,1}$ with a compact Calabi–Yau three-fold X. This is the so-called Calabi–Yau compactification which motivates the spaces we discuss here. Hertog–Horowitz–Maeda constructed classical configuration which has regions of negative energy density as seen from four-dimensional perspective [7]. This guides us to revisit the concept of the ADM mass (or the total energy) in string theory. A positive mass theorem for such spaces was established by Dai [4] and its Lorentzian version was discussed in [5].

In this short note, we formulate and prove a generalized positive energy theorem for spaces with asymptotic SUSY compactification which involves non-symmetric initial data.

We consider the complete Riemannian manifold (M^n, g_{ab}, p_{ab}) with non-symmetric data p_{ab}. Suppose $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$ for some $R > 0$ and X a compact simply connected Calabi–Yau manifold. We will call (M^n, g_{ab}, p_{ab}) a space with asymptotic SUSY compactification if the metric on the end M_∞ satisfies the following asymptotic conditions

$$g = \mathring{g} + h, \qquad \mathring{g} = g_{\mathbb{R}^k} + g_X, \tag{1.1}$$

$$h = O(r^{-\tau}), \qquad \overset{\circ}{\nabla} h = O(r^{-\tau-1}), \qquad \overset{\circ}{\nabla}\overset{\circ}{\nabla} h = O(r^{-\tau-2}), \tag{1.2}$$

and

$$p = O(r^{-\tau-1}), \qquad \overset{\circ}{\nabla} p = O(r^{-\tau-2}), \tag{1.3}$$

where p_{ab} is an arbitrary two-tensor satisfying $p_{\beta\alpha} = p_{\beta i} = p_{i\beta} = 0$, $\overset{\circ}{\nabla}$ is the Levi–Civita connection with respect to $\mathring{g}$, $\tau > 0$ is the asymptotic order, r is the Euclidean distance to a base point, and the index α, β run over the compact factor while the index i runs over the Euclidean part.

For such a space (M^n, g_{ab}, p_{ab}), the total energy is defined as

$$E = \lim_{R\to\infty} \frac{1}{4\omega_k \operatorname{vol}(X)} \int_{S_R\times X} (\partial_i g_{ij} - \partial_j g_{aa}) * \mathrm{d}x_j \,\mathrm{d}\operatorname{vol}(X), \tag{1.4}$$

and the total momentum is defined as

$$P_k = \lim_{R\to\infty} \frac{1}{4\omega_k \operatorname{vol}(X)} \int_{S_R\times X} 2(p_{kj} - \delta_{kj} p_{ii}) * \mathrm{d}x_j \,\mathrm{d}\operatorname{vol}(X). \tag{1.5}$$

Here the $*$ operator is the one on the Euclidean factor, the index i, j run over the Euclidean factor while the index a, b run over the full index of the manifold.

We say that (M^n, g_{ab}, p_{ab}) satisfies the dominant energy condition if

$$\mu \geq \max\left\{\sqrt{\sum_a (\omega_a)^2}, \qquad \sqrt{\sum_a (\omega_a + \chi_a)^2}\right\} + \sqrt{\sum_{1\leq a\leq n-3} \kappa_a^2}. \tag{1.6}$$

Here, local energy density is defined as

$$\mu = \frac{1}{2}\left(R + \left(\sum_a p_{aa}\right)^2 - \sum_{a,b} p_{ab}^2\right), \tag{1.7}$$

where R is the scalar curvature, and local momentum densities are defined as

$$\omega_a = \sum_b (\nabla_b p_{ab} - \nabla_a p_{bb}), \tag{1.8}$$

$$\chi_a = 2\sum_b \nabla_b \tilde{p}_{ba}, \tag{1.9}$$

$$\kappa_a^2 = \sum_{b,c,d;c>d>b>a} (\tilde{p}_{ab}\tilde{p}_{cd} + \tilde{p}_{ac}\tilde{p}_{db} + \tilde{p}_{ad}\tilde{p}_{bc})^2, \tag{1.10}$$

where $\tilde{p}_{ab} = p_{ab} - p_{ba}$.

Our main result is the following theorem.

Main theorem. *Let (M^n, g_{ab}, p_{ab}) be a complete spin manifold as above and the asymptotic order $\tau > (k-2)/2$ and $k \geq 3$. If (M^n, g_{ab}, p_{ab}) satisfies the dominant energy condition (1.6), then one has*

$$E \geq |P|. \tag{1.11}$$

Moreover, if $E = 0$ and $k = n$, then the following equation holds on M

$$\begin{aligned}
&\sum_{c<d}(R_{abcd} + p_{ac}p_{bd} - p_{ad}p_{bc})e^c e^d - \sqrt{-1}\sum_c (\nabla_a p_{bc} - \nabla_b p_{ac})e^c \\
&\quad = -\sqrt{-1}\left(\sum_{c,d;a\neq c\neq d\neq b\neq a} \nabla_a p_{cd} e^b e^c e^d - \sum_{c,d;a\neq c\neq d\neq b\neq a} \nabla_b p_{cd} e^a e^c e^d\right) \\
&\qquad - \left(\sum_{f,c,d;a\neq f\neq c\neq d\neq b\neq a} p_{cd} p_{af} e^b e^f e^c e^d - \sum_{f,c,d;a\neq f\neq c\neq d\neq b\neq a} p_{cd} p_{bf} e^a e^f e^c e^d\right)
\end{aligned} \tag{1.12}$$

as an endomorphism of the spinor bundle S, where R_{abcd} is the Riemann curvature tensor of the manifold (M^n, g_{ab}, p_{ab}).

Remarks.

1. This theorem extends without change to the case of X with any other special holonomy except $Sp(m) \cdot SP(1)$.

2. In particular, if the data p_{ab} is symmetric, then this theorem reduces to the result in [5].
3. This theorem corresponds to the result in [16] in the asymptotically flat case.

2. The Bochner–Lichnerowicz–Weitzenbock formula

Our argument is inspired by Witten [14,9]. We will adapt the spinor method [16,4,5] to our situation. The crucial point is that we use the Dirac-Witten operator $\tilde{D}$ which is defined in [16]. Our positive energy theorem is a consequence of a nice generalized Bochner–Lichnerowicz–Weitzenbock formula.

Fix a point $p \in M$ and an orthonormal basis $\{e_a\}$ of T_pM such that $(\nabla_a e_b)_p = 0$, where ∇ is the Levi–Civita connection of M. Let $\{e^a\}$ be the dual frame. Let S be the spinor bundle of M with Hermitian metric $\langle \cdot, \cdot \rangle$. The connection ∇ of M induces a connection on S. Define the modified connections $\tilde{\nabla}$ and $\bar{\nabla}$ on S as

$$\tilde{\nabla}_a = \nabla_a + \frac{\sqrt{-1}}{2} \sum_b p_{ab} e^b, \tag{2.1}$$

$$\bar{\nabla}_a = \nabla_a + \frac{\sqrt{-1}}{2} \sum_b p_{ab} e^b - \frac{\sqrt{-1}}{2} \sum_{b,c;a \neq b \neq c \neq a} p_{bc} e^a e^b e^c. \tag{2.2}$$

Then the Dirac operator D and the Dirac-Witten operator $\tilde{D}$ are defined as

$$D = \sum_a e^a \nabla_a, \tag{2.3}$$

$$\tilde{D} = \sum_a e^a \tilde{\nabla}_a, \tag{2.4}$$

respectively. Moreover, we have the following formulae:

$$\mathrm{d}(\langle \phi, \psi \rangle \mathrm{int}(e^a) \mathrm{d\,vol}) = \left(\langle \tilde{\nabla}_a \phi, \psi \rangle + \left\langle \phi, \left(\tilde{\nabla}_a - \sqrt{-1} \sum_b p_{ab} e^b \right) \psi \right\rangle \right) \mathrm{d\,vol} \tag{2.5}$$

$$= \left(\langle \bar{\nabla}_a \phi, \psi \rangle + \left\langle \phi, \left(\bar{\nabla}_a - \sqrt{-1} \sum_b p_{ab} e^b \right) \psi \right\rangle \right) \mathrm{d\,vol}, \tag{2.6}$$

$$\mathrm{d}(\langle e^a \phi, \psi \rangle \mathrm{int}(e^a) \mathrm{d\,vol}) = \left(\langle \tilde{D} \phi, \psi \rangle - \left\langle \phi, \left(\tilde{D} + \sqrt{-1} \sum_a p_{aa} \right) \psi \right\rangle \right) \mathrm{d\,vol}. \tag{2.7}$$

We denote the adjoint operators by

$$\tilde{\nabla}_a^* = -\tilde{\nabla}_a + \sqrt{-1} \sum_b p_{ab} e^b, \tag{2.8}$$

$$\bar{\nabla}_a^* = -\bar{\nabla}_a + \sqrt{-1} \sum_b p_{ab} e^b, \tag{2.9}$$

$$\tilde{D}^* = \tilde{D} + \sqrt{-1}\sum_a p_{aa}. \tag{2.10}$$

Now we recall two nice formulae in [16].

Proposition 2.1. *One has*

$$\tilde{D}^*\tilde{D} = \bar{\nabla}^*\bar{\nabla} + \frac{1}{2}\left(\mu + \sqrt{-1}\sum_b \omega_b e^b\right) + \frac{1}{2}\mathcal{F}, \tag{2.11}$$

$$\tilde{D}\tilde{D}^* = \bar{\nabla}\bar{\nabla}^* + \frac{1}{2}\left(\mu - \sqrt{-1}\sum_b (\omega_b + \chi_b) e^b\right) - \frac{1}{2}\mathcal{F}, \tag{2.12}$$

where $\mathcal{F} = \sum_{a\neq b\neq c\neq d\neq a} p_{ab}p_{cd}e^ae^be^ce^d$.

We are going to derive the integral form of the generalized Bochner–Lichnerowicz–Weitzenbock formula.

Lemma 2.1. *One has*

$$\int_{\partial M}\langle\phi, \bar{\nabla}_a\phi + e^a\tilde{D}\phi\rangle \mathrm{int}(e^a) d\,\mathrm{vol}(g)$$
$$= \int_M |\bar{\nabla}\phi|^2 + \frac{1}{2}\left\langle \phi, \left(\mu + \sqrt{-1}\sum_a \omega_a e^a\right)\phi\right\rangle + \int_M \frac{1}{2}\langle\phi, \mathcal{F}\phi\rangle - |\tilde{D}\phi|^2. \tag{2.13}$$

Proof. By (2.11),

$$\mathrm{RHS} = \int_M |\bar{\nabla}\phi|^2 + \langle\phi, \tilde{D}^*\tilde{D}\phi\rangle - |\tilde{D}\phi|^2 - \langle\phi, \bar{\nabla}^*\bar{\nabla}\phi\rangle$$
$$= \int_{\partial M}\langle\phi, \bar{\nabla}_a\phi\rangle \mathrm{int}(e^a)\mathrm{d}\,\mathrm{vol}(g) - \int_{\partial M}\langle e^a\phi, \tilde{D}\phi\rangle \mathrm{int}(e^a)\mathrm{d}\,\mathrm{vol}(g) = \mathrm{LHS}.$$

3. Manifolds with parallel spinors

Recall that the spin manifold $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$ for some $R > 0$. Since $k \geq 3$ and X is simply connected, the end M_∞ is also simply connected and therefore has a unique spin structure which comes from the product of the restriction of the spin structure on $\mathbb{R}^k$ and the spin structure on X. One has the following result in [13].

Proposition 3.1. *Let* (M, g) *be a complete, simply connected, irreducible Riemannian spin manifold and N be the dimension of parallel spinors. Then* $N > 0$ *if and only if the holonomy group of M is one of* $SU(m)$, $Sp(m)$, *Spin(7)*, G_2.

Remark. In physics language, manifolds with parallel spinors are said to be supersymmetric (SUSY).

We denote by $\{e_a^0\}$ the orthonormal basis of $\mathring{g}$ which consists of $\partial/\partial x^i$ followed by an orthonormal basis $\{f_\alpha\}$ of g_X. Orthonormaling the orthonormal frame $\{e_a^0\}$ with respect to $\mathring{g}$ yields an orthonormal frame $\{e_a\}$ with respect to g. Moreover,

$$e_a = e_a^0 - \tfrac{1}{2} h_{ab} e_b^0 + O(r^{-2\tau}). \tag{3.1}$$

This provides a gauge transformation $\mathcal{A}$ of the tangent bundles on the end M_∞:

$$\mathcal{A} : SO(\mathring{g}) \to SO(g),$$

$$e_a^0 \mapsto e_a.$$

Hence it induces a map from the spinor bundles.

Now we pick a unit norm parallel spinor ψ_0 of $(\mathbb{R}^k, g_{\mathbb{R}^k})$ and a unit parallel spinor ψ_1 of (X, g_X). Then $\phi_0 = \mathcal{A}(\psi_0 \otimes \psi_1)$ defines a spinor of M_∞. We extend ϕ_0 smoothly inside and note that

$$\nabla \phi_0 = O(r^{-\tau-1}) \tag{3.2}$$

which is a consequence of an asymptotic formula in [4].

4. Fibred boundary calculus and the Dirac-Witten equation

In this section, we will use the fibred boundary calculus of Melrose–Mazzeo [8] to solve the Dirac-Witten equation. The argument is following Dai's [4].

Let $\bar{M}$ be a smooth compact manifold with boundary and suppose that x is a boundary defining function such that x vanishes on $\partial\bar{M}$ and $\mathrm{d}x \neq 0$ there. Assume further that the boundary $\partial\bar{M}$ comes with a fibration structure $F \to \partial\bar{M} \xrightarrow{\pi} B$ with fiber F. Then the metric g is called a fibred boundary metric if in a neighborhood of the boundary $\partial\bar{M}$, the metric g takes the form

$$g = \frac{\mathrm{d}x^2}{x^4} + \frac{\pi^*(g_{\mathrm{B}})}{x^2} + g_{\mathrm{F}}, \tag{4.1}$$

where g_{B} is a metric on the base B and g_{F} is a family of fiberwise metrics.

In the setting of spaces with asymptotic SUSY compactification, the change of variable $x = 1/r$ gives a trivial fibration $S^{k-1} \times X$.

Sometimes we use the notation M and $\bar{M}$ interchangeably. For a manifold with boundary, we introduce two Lie algebras of vector fields:

- b-vector fields

$$\mathcal{V}_b(M) := \{V | V \text{ tangent to the boudnary} \partial M\}, \tag{4.2}$$

- fibred boundary vector fields

$$\mathcal{V}_{fb}(M) := \{V \in \mathcal{V}_b(M) | V \text{ tangent to the fiber } F \text{ at } \partial M, \quad Vx = O(x^2)\}. \tag{4.3}$$

The Sobolev space $L^{p,2}(M, S)$ is defined as

$$L^{p,2}(M, S) := \{\phi \in L^2(M, S) | \nabla_{V_1} \cdots \nabla_{V_j}\phi \in L^2(M, S), \forall j \leq p, \quad V_i \in \mathcal{V}_b(M)\}. \tag{4.4}$$

Let $\gamma \in \mathbb{R}$ and we define the space of conormal sections of order γ by

$$\mathcal{A}^\gamma(M, S) := \{\phi \in C^\infty(M, S) | |\nabla_{V_1} \cdots \nabla_{V_j}\phi| \leq Cx^\gamma, \quad \forall j, \quad V_i \in \mathcal{V}_b(M)\}, \tag{4.5}$$

and the subspace of polyhomogeneous sections by

$$\mathcal{A}^*_{\text{phg}}(M, S) := \left\{ \phi \in \mathcal{A}^*(M, S) | \phi \sim \sum_{Re\gamma_j \to \infty} \sum_{k=0}^{N_j} \psi_{jk} x^{\gamma_j} (\log x)^k, \quad \psi_{jk} \in C^\infty(\partial M, S) \right\}. \tag{4.6}$$

These expansions are meant in the usual asymptotic sense as $x \to 0$ and hold along with all derivatives. The superscript $*$ may be replaced by an index set I containing all pairs (γ_j, N_j) which appear in this expansion.

Denote by $\Pi_0 : L^2(M, S) \to \text{Ker} D_F$ the natural orthogonal projector and let $\Pi_\perp := Id - \Pi_0$.

The following proposition is a summary of the results in [6] (see also [4], Theorem 3.1).

Proposition 4.1. Suppose that a is not an indicial root of $\Pi_0 x^{-1} D \Pi_0$. *Then*

$$D : x^a L^{1,2}(M, S) \to x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$$

is Fredholm. If $D\phi = 0$ for $\phi \in x^a L^2(M, S)$, then ϕ is polyhomogeneous with exponents in its expansion determined by the indicial roots of $\Pi_0 x^{-1} D \Pi_0$ and truncated at a. If $D\xi = \psi$ for $\psi \in \mathcal{A}^a(M, S)$ and $\xi \in x^{c-1}\Pi_0 L^{1,2}(M, S) \oplus x^c \Pi_\perp L^{1,2}(M, S)$ and $c < a$, then $\xi \in \Pi_0 \mathcal{A}^I_{\text{phg}}(M, S) + \mathcal{A}^a(M, S)$.

Remarks.

1. Strictly speaking, only the metric $\mathring{g}$ is a fibred boundary metric. However, it is easy to see that the results generalize to the metric g (see [4]). The metric perturbation produces only a lower order term.
2. In our situation, note that $\tilde{D} = D + (\sqrt{-1}/2)\sum_{a,b} p_{ab} e^a e^b = D + O(r^{-\tau-1})$. It follows from the decay condition of the initial data p_{ab} that the Dirac-Witten operator $\tilde{D}$ is also a Fredholm operator from $x^a L^{1,2}(M, S)$ to $x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$.
3. The precise forms of these results for the Dirac-Witten operators $\tilde{D}$ and $\tilde{D}^*$ are somewhat different, but one still has the regularity property.
4. For the precise definition of the indicial root, we refer to [8,6]. For our purpose, we only note that the set of indicial roots is discrete.

To prove that the Dirac-Witten operator $\tilde{D}$ is an isomorphism under certain conditions, we need the following lemma inspired by [9,16].

Lemma 4.1. *Suppose* $(M^n, g_{\underline{ab}}, p_{ab})$ *is a complete spin manifold as above and the spinor* ϕ *satisfying either* $\overline{\nabla}\phi = 0$ *or* $\overline{\nabla}^*\phi = 0$. *If* $\lim_{r\to\infty}\phi = 0$, *then* $\phi = 0$.

Proof. By the assumptions, we have $|d|\phi|^2| = 2|\langle Re\nabla\phi, \phi\rangle| \le C|p||\phi|^2$, where C is some constant. This implies $|d\ \log|\phi|| \le Cr^{-\tau-1}$ outside a compact set. Fix a point (r_0, y_0) and integrate along a path from (r_0, y_0) with respect to r. Then one has

$$|\phi(r, y_0)| \ge |\phi(r_0, y_0)|e^{C(r_0^{-\tau}-r^{-\tau})}.$$

Taking $r \to \infty$ or taking (r, y_0) to be the zero of ϕ, we get $\phi(r_0, y_0) = 0$. Hence $\phi = 0$ when r is large enough. It follows from the unique continuation property that $\phi = 0$ since ϕ satisfies the Dirac-Witten equation. We complete the proof of this lemma.

Lemma 4.2. *If the dominant energy condition* (1.6) *holds and* $a > (k-2)/2$ *is not an indicial root, then*

$$\tilde{D} : x^a L^{1,2}(M, S) \to x^{a+1}\Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$$

is an isomorphism.

Proof. The argument here is similar to Dai's (see [4, Section 3]). We first see that $\tilde{D}$ is injective. If $\phi \in \mathrm{Ker}\,\tilde{D} \subset x^a L^{1,2}(M, S)$, then by elliptic regularity, $\phi \in \mathcal{A}^a_{\mathrm{phg}}(M, S)$. By the Weitzenbock formula (2.13)

$$\int_\Omega \left\{ |\overline{\nabla}\phi|^2 + \frac{1}{2}\left\langle \phi, \left(\mu + \sqrt{-1}\sum_a \omega_a e^a\right)\phi \right\rangle + \frac{1}{2}\langle\phi, \mathcal{F}\phi\rangle \right\} \mathrm{d\,vol}$$
$$= \int_{\partial\Omega} \langle \phi, \overline{\nabla}_a\phi + e^a\tilde{D}\phi\rangle \mathrm{int}(e^a)\mathrm{d\,vol}.$$

By taking Ω so that $\partial\Omega = S_r \times X$ and $r \to \infty$ we see that the right hand side of the above equality tends to zero since $\phi \in \mathcal{A}^a_{\mathrm{phg}}(M, S)$ and $a > (k-2)/2$. It follows from the dominant energy condition (1.6) that $\overline{\nabla}\phi = 0$ and hence $\phi = 0$ by Lemma 4.1.

The same argument as above applies to the adjoint operator $\tilde{D}^*$. By the Fredholm property, the surjectivity of $\tilde{D}$ follows from the injectivity of $\tilde{D}^*$ which is a consequence of the Weitzenbock formula (2.12) as well as Lemma 4.1. This proves the lemma.

Now we are ready to solve the Dirac-Witten equation.

Lemma 4.3. *There exists a smooth spinor* ϕ *such that* $\tilde{D}\phi = 0$ *and* $\phi = \phi_0 + O(r^{-\tau})$.

Proof. We construct the wanted spinor by setting $\phi = \phi_0 + \xi$ and solve $\tilde{D}\xi = -\tilde{D}\phi_0 = O(r^{-\tau-1})$. By Lemma 4.2, adjusting τ slightly if necessary so that it is not one of the indicial root, we have a solution $\xi = O(r^{-\tau})$.

5. Proof of the main theorem

Lemma 5.1. *If a spinor* ϕ *is asymptotic to* $\phi_0 : \phi = \phi_0 + O(r^{-\tau})$*, then one has*

$$\lim_{R\to\infty}\int_{S_R\times X}\langle\phi, \bar{\nabla}_a\phi + e^a\tilde{D}\phi\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g) = \omega_k\,\mathrm{vol}(X)\langle\phi_0, E\phi_0 + \sqrt{-1}P_i\,\mathrm{d}x^i\phi_0\rangle. \tag{5.1}$$

Proof.

$$\begin{aligned}
&\int_{S_R\times X}\langle\phi, \bar{\nabla}_a\phi + e^a\tilde{D}\phi\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g)\\
&= \int_{S_R\times X}\left\langle\phi, \nabla_a + \frac{\sqrt{-1}}{2}\sum_b p_{ab}e^b - \frac{\sqrt{-1}}{2}\sum_{b,c;a\neq b\neq c\neq a} p_{bc}e^ae^be^c\phi\right\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g)\\
&\quad + \int_{S_R\times X}\left\langle\phi, e^a\sum_b e^b\left(\nabla_b + \frac{\sqrt{-1}}{2}\sum_c p_{bc}e^c\right)\phi\right\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g),\\
&= \int_{S_R\times X}\langle\phi, \nabla_a\phi + e^aD\phi\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(X)\\
&\quad + \int_{S_R\times X}\left\langle\phi, \frac{\sqrt{-1}}{2}\left(\sum_b p_{ab}e^b - \sum_{b,c;a\neq b\neq c\neq a} p_{bc}e^ae^be^c + \sum_{b,c} p_{bc}e^ae^be^c\right)\phi\right\rangle\\
&\quad \times \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g).
\end{aligned} \tag{5.2}$$

The first term in (5.2) is computed in [4] which tends to $\omega_k\,\mathrm{vol}(X) < \phi_0, E\phi_0 >$ as $r\to\infty$.

The second term is

$$\begin{aligned}
&\int_{S_R\times X}\left\langle\phi, \frac{\sqrt{-1}}{2}\left(\sum_b p_{ab}e^b + \left(\sum_{a=b;b\neq c} + \sum_{a=c;b\neq c} + \sum_{b=c}\right) p_{bc}e^ae^be^c\right)\phi\right\rangle\\
&\times \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g)\\
&\quad = \int_{S_R\times X}\left\langle\phi, \frac{\sqrt{-1}}{2}\left(\sum_b p_{ab}e^b + \sum_{b\neq a} p_{ab}e^ae^ae^b + \sum_{b\neq a} p_{ba}e^ae^be^a\right.\right.\\
&\qquad \left.\left. + \sum_b p_{bb}e^ae^be^b\right)\phi\right\rangle \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g)\\
&\quad = \int_{S_R\times X}\left\langle\phi, \frac{\sqrt{-1}}{2}\left(\sum_b p_{ab}e^b - \sum_{b\neq a} p_{ab}e^b + \sum_{b\neq a} p_{ba}e^b - \sum_c p_{cc}e^a\right)\phi\right\rangle\\
&\qquad \times \mathrm{int}(e_a)\mathrm{d}\,\mathrm{vol}(g)
\end{aligned}$$

$$= \int_{S_R \times X} \left\langle \phi, \frac{\sqrt{-1}}{2} \left(\sum_b p_{ba} e^b - \delta_{ba} p_{cc} e^b \right) \phi \right\rangle \mathrm{int}(e_a) \mathrm{d}\,\mathrm{vol}(g)$$

which goes to $\omega_k \,\mathrm{vol}(X) \langle \phi_0, \sqrt{-1} P_i \,\mathrm{d}x^i \phi_0 \rangle$ as $r \to \infty$.

Proof of the main theorem. Now we are ready to prove our main result. Note that $\sqrt{-1} P_i \,\mathrm{d}x^i$ has eigenvalues $\pm|P|$. We take ϕ_0 as the unit eigenspinor of eigenvalue $-|P|$. It follows from the Weitzenbock formula (2.13) that

$$E \geq |P|.$$

The proof of the second part is the same as in [16].

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